THE GROWTH RATES OF NON-COCOMPACT 3-DIMENSIONAL HYPERBOLIC COXETER TETRAHEDRA AND PYRAMIDS

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Abstract. We determine the growth functions of Coxeter groups with respect to non-compact hyperbolic Coxeter tetrahedra and pyramids and show that the growth rates of them are Perron numbers. More details will be presented in our forthcoming paper [6].

1. Introduction

1.1. The growth series and the growth rate. Let \((G, S)\) be the pair of a group \(G\) with a finite generating set \(S\) satisfying \(S = S^{-1}\) and \(1_G \not\in S\). Then we can define the word length \(\ell_S(g)\) of \(g \in G\) with respect to \(S\) by the smallest integer \(n \geq 0\) for which there exist \(s_1, s_2, \ldots, s_n \in S\) such that \(g = s_1s_2 \cdots s_n\). The growth series of \((G, S)\) is defined by the power series \(f_S(t) = \sum_{k \geq 0} a_k t^k\), where \(a_k\) is the number of elements \(g \in G\) satisfying \(\ell_S(g) = k\).

If we consider \(t \in \mathbb{C}\), the growth series always has positive radius of convergence since \(a_k < |S|^k\), where \(|S|\) denotes the cardinality of \(S\). If \(G\) is an infinite group then the radius of convergence is at most one. In this case we call the reciprocal of the radius of convergence the growth rate of \((G, S)\) and denote it by \(\tau\).

1.2. Coxeter groups and their growth functions. Let \(X^n\) denote one of the metric \(n\)-spaces of constant curvature, the spherical space \(S^n\), the Euclidean space \(E^n\) or the hyperbolic space \(H^n\). A convex polytope \(P \subset X^n\) of finite volume is called a Coxeter polytope if its dihedral angles are submultiples of \(\pi\) or 0.

Let \(G\) be a discrete group generated by finitely many reflections in hyper-planes (mirrors) containing the facets of a Coxeter polytope \(P\) and denote by \(S\) the (natural) set of generating reflections. Then we call \(G\) a geometric Coxeter group with respect to \(P\). Note that the Coxeter polytope \(P\) itself is a fundamental domain of the group \(G\). For each generator \(s \in S\), one has \(s^2 = 1\) while two distinct elements \(s, s' \in S\) satisfy either no relation or provide the relation \((ss')^m = 1\) for an integer \(m = m(s, s') \geq 2\). The first case arises if the corresponding mirrors admit a common perpendicular or intersect in the boundary of \(X^n\), while the second case arises if the mirrors intersect in \(X^n\).

For Coxeter groups, we use the standard notation by means of the associated Coxeter graph. In particular, two nodes in the Coxeter graph of \(G\) corresponding to mirrors intersecting under the angle of \(\pi/3\) (respectively \(\pi/p\))
are connected by a simple edge (respectively by an edge with label $p$). If two mirrors are perpendicular, their nodes are not joined at all. If two mirrors intersect in the boundary of $X^n$, the nodes are joined by a bold edge ([11]).

In this work we consider mostly $X^n = \mathbb{H}^n$, and in that case we call a Coxeter group $(G, S)$ a hyperbolic Coxeter group.

From now on, we consider the growth series of the Coxeter group $(G, S)$, where $S$ is its natural set of generating reflections. In practice, the growth series $f_S(t)$, which is analytic on $|t| < R$, where $R$ is the radius of convergence of $f_S(t)$, extends to a rational function $P(t)/Q(t)$ on $\mathbb{C}$ by analytic continuation, where $P(t), Q(t) \in \mathbb{Z}[t]$ are relatively prime. There are formulas due to Solomon and Steinberg to calculate the rational function $P(t)/Q(t)$ from the Coxeter graph of $(G, S)$ ([9, 10]; see also [4]), and we call this rational function the growth function of $(G, S)$.

**Theorem 1.** (Solomon’s formula)
The growth function $f_S(t)$ of an irreducible finite Coxeter group $(G, S)$ can be written as $f_S(t) = \prod_{i=1}^{k} [m_i + 1]$, where $[n] := 1 + t + \cdots + t^{n-1}$ and $\{m_1, m_2, \cdots, m_k\}$ is the set of exponents of $(G, S)$.

We give explicitly the growth functions of irreducible finite Coxeter groups in Table 1, where we use the notation $[n, m] = [n][m]$ (see [4, 5]).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Exponents</th>
<th>$f_S(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$1, 2, \cdots, n$</td>
<td>$[2, 3, \cdots, n + 1]$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$1, 3, \cdots, 2n - 1$</td>
<td>$[2, 4, \cdots, 2n]$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$1, 3, \cdots, 2n - 3, n - 1$</td>
<td>$[2, 4, \cdots, 2n - 2][n]$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$1, 4, 5, 7, 8, 11$</td>
<td>$[2, 5, 6, 8, 9, 12]$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$1, 5, 7, 9, 11, 13, 17$</td>
<td>$[2, 6, 8, 10, 12, 14, 18]$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$1, 7, 11, 13, 17, 19, 23, 29$</td>
<td>$[2, 8, 12, 14, 18, 20, 24, 30]$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$1, 5, 7, 11$</td>
<td>$[2, 6, 8, 12]$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$1, 5, 9$</td>
<td>$[2, 6, 10]$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>$1, 11, 19, 29$</td>
<td>$[2, 12, 20, 30]$</td>
</tr>
<tr>
<td>$I_2(m)$</td>
<td>$1, m - 1$</td>
<td>$[2, m]$</td>
</tr>
</tbody>
</table>

**Table 1.** The growth functions of irreducible finite Coxeter groups

**Theorem 2.** (Steinberg’s formula)
Let $(G, S)$ be an infinite Coxeter group. Let us denote by $(G_T, T)$ the Coxeter subgroup of $(G, S)$ generated by the subset $T \subseteq S$, and let its growth function be $f_T(t)$. Set $\mathcal{F} = \{ T \subseteq S : G_T \text{ is finite} \}$. Then
\[
\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} (-1)^{|T|} f_T(t).
\]

1.3. Growth rates of Coxeter groups. It is known that the growth rates of Coxeter groups are bigger than 1 ([3]). In this case, the growth function $f_S(t) = P(t)/Q(t)$ has a real pole $t = R$, where $R$ is the radius of convergence of the growth series $f_S(t)$. The results of Solomon and Steinberg imply that
$P(0) = 1$, and hence $a_0 = 1$ means that $Q(0) = 1$. Therefore the growth rate $\tau > 1$ becomes a real algebraic integer. If $t = R$ is the unique pole of $f_S(t) = P(t)/Q(t)$ on the circle $|t| = R$, then $\tau$ is a real algebraic integer all of whose conjugates have strictly smaller absolute values. Such a real algebraic integer is called a *Perron number*.

For two and three-dimensional cocompact hyperbolic Coxeter groups (having a compact fundamental polytope), Cannon-Wagreich and Parry showed that the growth rates are Salem numbers ([1, 7]), where a real algebraic integer $\tau > 1$ is called a *Salem number* if $\tau^{-1}$ is an algebraic conjugate of $\tau$ and all algebraic conjugates of $\tau$ other than $\tau$ and $\tau^{-1}$ lie on the unit circle. It follows from the definition that a Salem number is a Perron number.

Kellerhals and Perren calculated the growth functions of all four-dimensional cocompact hyperbolic Coxeter groups with at most 6 generators and showed that $\tau$ are not Salem numbers anymore while they checked numerically that $\tau$ are Perron numbers ([5]).

In the non-cocompact case, Floyd proved that the growth rates of two-dimensional hyperbolic Coxeter groups are *Pisot-Vijayaraghavan numbers*, where a real algebraic integer $\tau > 1$ is called a Pisot-Vijayaraghavan number if all algebraic conjugates of $\tau$ other than $\tau$ lie in the open unit disk ([2]). A Pisot-Vijayaraghavan number is also a Perron number by definition.

From these results for low-dimensional cases, Kellerhals and Perren conjectured that the growth rates of hyperbolic Coxeter groups are always Perron numbers. In the present paper, we consider three-dimensional non-cocompact hyperbolic Coxeter groups with 4 and 5 generators. They were classified by Lannér ([4, 8]) (see Table 2) and by Tumarkin ([11]) (see Table 3) respectively. We will show that the growth rates of them are Perron numbers.
Theorem 2. Consider the diagram $\circ$ in Table 2.

There are two non-isomorphic Coxeter subgroups with three generators and two non-isomorphic Coxeter subgroups with two generators. Their corresponding growth functions are given by Solomon’s result (cf. Theorem 1) as follows.

<table>
<thead>
<tr>
<th>type of subgroup</th>
<th>growth function</th>
<th>number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_3$</td>
<td>$[2, 3, 4]$</td>
<td>2</td>
</tr>
<tr>
<td>$A_2 \times A_1$</td>
<td>$[2, 2, 3]$</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$[2, 3]$</td>
<td>4</td>
</tr>
<tr>
<td>$A_1 \times A_1$</td>
<td>$[2, 2]$</td>
<td>2</td>
</tr>
</tbody>
</table>
There are also four Coxeter subgroups with 1 generator, and therefore of type $A_1$. By Theorem 2, we have that
\[ \frac{1}{f_S(t^{-1})} = -2 + \frac{-1}{[2, 2, 3]} + \frac{4}{[2, 2]} + \frac{2}{2} + \frac{-4}{[2]} + 1. \]
This yields the growth function
\[ f_S(t) = \frac{(t + 1)(t^2 + 1)(t^2 + t + 1)}{(t - 1)(t^3 + t - 1)}. \]

The denominator polynomials $Q(t)$ of the growth functions $P(t)/Q(t)$ for the other three-dimensional non-cocompact hyperbolic Coxeter groups with 4 generators are given below. (The numbers of polynomials correspond to the numbers of graphs in Table 2.)

- (2) $t(t^4 + t^3 + t^2 + t - 1)$
- (3) $t(t^4 + t^3 + t^2 + t - 1)$
- (4) $t(t^7 + t^6 + 2t^5 + 2t^4 + t^3 + t^2 - 1)$
- (5) $t(t^9 + t^7 + t^6 + t^4 + t^2 + t - 1)$
- (6) $t(t^4 + t^2 + t - 1)$
- (7) $t(t^7 + t^6 + t^5 + t^4 + t^3 - 1)$
- (8) $t(t^2 + t - 1)(t^2 + t + 1)$
- (9) $t(t^5 + t^3 + t - 1)$
- (10) $t(t^4 + 3t^3 + t^2 - 1)$
- (11) $t(t^5 + t^4 + t^2 + t - 1)$
- (12) $t(t^6 + t^5 + t^4 + t^3 + t^2 + t - 1)$
- (13) $t(3t^3 + t^2 + t - 1)$
- (14) $t(t^5 + t^4 + t - 1)$
- (15) $t(t^8 + 2t^7 + 2t^6 + 3t^5 + t^4 + t^3 - 1)$
- (16) $t(t^7 + t^6 + t^5 + t^4 - 1)$
- (17) $t(t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1)$
- (18) $t(t^{13} + t^{12} + t^{11} + 2t^9 + 2t^8 + 2t^7 + 2t^6 + 2t^5 + t^4 + t^3 - 1)$
- (19) $t(t^2 + t - 1)(t^4 + t^3 + t^2 + t + 1)$
- (20) $t(t^7 + t^6 + 2t^5 + t^4 + t^3 + t - 1)$
- (21) $t(t^4 + t^3 + t^2 + t - 1)(t^4 + t^3 + t^2 + t + 1)$
- (22) $t(t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t - 1)$
- (23) $t(t^5 + t^4 + t^3 + t^2 + t - 1)$

The denominator polynomials of the growth functions for three-dimensional non-cocompact hyperbolic Coxeter groups with five generators are given below, where $(k, l, m, n)$ or $(k, l, m)$ represents the Coxeter graph in Table 3.

- $(2, 2, 2, 2, 2)$ : $(t - 1)(t^5 + 2t^4 + 2t^3 + t^2 - 1)$
- $(2, 2, 2, 2, 3)$ : $(t - 1)(t^7 + t^6 + 2t^5 + t^4 + 2t^3 + t - 1)$
- $(2, 2, 3, 3, 3)$ : $(t - 1)(t^4 + 2t^3 + t^2 + t - 1)$
- $(2, 2, 3, 3, 4)$ : $(t - 1)(t^7 + 2t^6 + 2t^5 + 2t^4 + 2t^3 + t^2 + t - 1)$
- $(2, 2, 3, 4, 4)$ : $(t - 1)(t^5 + t^4 + t^3 + 2t - 1)$
- $(2, 4, 2, 2, 4)$ : $(t - 1)(t^4 + 2t^3 + t^2 + t - 1)$
- $(2, 4, 2, 3, 3)$ : $(t - 1)(t^7 + 2t^6 + 2t^5 + 3t^4 + 2t^3 + t^2 + t - 1)$
- $(2, 4, 3, 3, 4)$ : $(t - 1)(t^5 + 2t^7 + 3t^6 + 3t^5 + 3t^4 + 3t^3 + t^2 + t - 1)$
- $(2, 4, 4, 4)$ : $(t - 1)(2t^4 + 3t^3 + 2t^2 + t - 1)$
Proof. Notice that all denominator polynomials have the form $Q(t) = (t - 1)(\sum_{k=1}^{n} b_k t^k - 1)$ and the greatest common divisor of $\{k \in \mathbb{N} | b_k \neq 0\}$ is 1. We consider two cases.

(Case 1) Let us consider the case except for the groups $(5, 3, 3)$, $(5, 4, 2)$, $(5, 4, 3)$, and $(5, 5, 3)$, where $b_k(k \geq 2)$ is a non-negative integer. Let us put $h(t) = \sum_{k=1}^{n} b_k t^k$.

(Step 1) Observe $h(0) = 0$, $h(1) > 1$, and $h(t)$ is strictly monotonically increasing on $(0, 1)$. By the intermediate value theorem, there exists a unique real number $r$ in the open interval $(0, 1)$ such that $h(r) = 1$, that is, $Q(r) = 0$.

(Step 2) By Step 1, $r$ is the radius of convergence of $f_S(t)$.
(Step 3) Now let us show that $f_S(t)$ has no poles on the circle $|t| = r$ other than $r$. Consider a complex number $re^{i\theta}$ on the circle $|t| = r$, where $0 \leq \theta < 2\pi$. If $Q(re^{i\theta}) = 0$, that is, $h(re^{i\theta}) = 1$, then this implies

$$1 = \sum_{k=1}^{n} b_k r^k \cos k\theta \leq \sum_{k=1}^{n} b_k r^k = 1.$$ 

Hence $\cos k\theta = 1$ for all $k \in \mathbb{N}$ with $b_k \neq 0$. The assumption that the greatest common divisor of $\{k \in \mathbb{N} \mid b_k \neq 0\}$ is 1 means that $\theta = 0$.

(Case 2) Let us consider the groups $(5, 3, 3)$, $(5, 4, 2)$, $(5, 4, 3)$, and $(5, 5, 3)$. Let us focus on the case $(5, 4, 3)$. (The other cases will be presented in our forthcoming paper [6].) Write $Q(t) = (t - 1) g(t)$ with $g(t) = t^9 + t^8 + 2t^6 + 3t^3 - 2t^2 + 3t - 1$.

(Step 1) Since $g'(t) = 9t^8 + 8t^7 + 12t^5 + 9(t - 2)^2 + 2\pi$, $g(t)$ has a unique real root $r$ in the open interval $(0, 1)$, which is therefore the radius of convergence of $f_S(t)$.

(Step 2) Now let us show that $f_S(t)$ has no poles on the circle $|t| = r$ other than $r$. Consider a complex number $re^{i\theta}$ on the circle $|t| = r$, where $0 \leq \theta < 2\pi$. If $Q(re^{i\theta}) = 0$, then $g(re^{i\theta}) = g(r) = 0$. By comparing real parts, we have

$$3(1 - \cos \theta) + 3r^2(1 - \cos 3\theta) + \cdots + r^8(1 - \cos 9\theta) = 2r(1 - \cos 2\theta).$$

Since $1 - \cos 2\theta > 0$ and $1 - \cos k\theta \geq 0$ for any integer $k$, we have

$$r \geq \frac{3(1 - \cos \theta)}{2(1 - \cos 2\theta)} = \frac{3}{8} \left(\frac{\theta}{2}\right)^2 \geq \frac{3}{8}.$$ 

As $g(t)$ is monotonically increasing on $(0, 1)$,

$$0 = g(r) > g\left(\frac{3}{8}\right) = \frac{1080811}{134217728} > 0,$$

which is a contradiction. □

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