1 Definitions

1.1 The growth function \( f_s(t) \) of \((G, S)\)

- \((G, S)\): a finite generated group and its generator set, 
  \(\#S < \infty, S = S^{-1}\),
- The length of \(g \in G\) with respect to \(S\) is defined by 
  \(l_S(g) := \min \{ n \in \mathbb{N} | g = s_1 \cdots s_n, s_i \in S \}\), \(l_S(1_G) := 0\).
- Let \(a_k := \#\{ g \in G | l_S(g) = k \}\).
- The growth series of \((G, S)\) is defined by the formal power series 
  \(f_s(t) := \sum_{k=0}^{\infty} a_k t^k\).
- The growth rate of \((G, S)\) is defined by 
  \(\tau := \limsup_{k \to \infty} \sqrt[k]{a_k} (= \frac{1}{R})\), 
  where \(R\) is the radius of convergence of \(f_s(t)\) if \(t \in \mathbb{C}\).
  \(1 \leq \tau \leq \#S\).

1.2 Geometric Coxeter groups

- \(X^n : H^n, E^n\) or \(S^n\)
  a Coxeter polyhedron \(P \subset X^n\): 
  a polyhedron with dihedral angles \(\frac{\pi}{p}\) where \(p \geq 2\) is an integer or \(\infty\)
  a Coxeter group \((G, S)\) generated by \(P\): 
  a discrete subgroup of \(Isom(X^n)\) generated by reflections \(S := \{s_1, \ldots, s_m\}\) with respects to the facets of \(P\).

2 Growth functions of geometric Coxeter groups

2.1 Growth functions of Coxeter groups

Theorem 1. (Solomon’s formula)

\((G, S)\): a finite (geometric) Coxeter group
Then the growth function \(f_s(t)\) of \((G, S)\) can be written as
\[f_s(t) = \prod_{k=1}^{m} (m_1 + 1)\]
where \([n] := 1 + t + \cdots + t^{n-1}\) and \(\{m_1, m_2, \ldots, m_k\}\) is the set of exponents of \((G, S)\).

Theorem 2. (Steinberg’s formula)

\((G, S)\): an infinite (geometric) Coxeter group
\((G_T, T)\): a Coxeter subgroup of \((G, S)\) generated by the subset \(T \subset S\)
\(f_s(t)\): the growth function of \((G, S)\)
\(f_T(t)\): the growth function of \((G_T, T)\)
\(F := \{T \subset S : G_T \text{ is finite}\}\)

Then
\[\frac{1}{f_s(t)^{|T|}} = \sum_{T \in F} (-1)^{|T|} f_T(t)\]

(2.3 Former results and the main theorem)

\(P\): a Coxeter polyhedron in \(\mathbb{H}^n\)
\((G, S)\): a Coxeter group generated by \(P\)

The arithmetic property of the growth rate \(\tau(= 1/R)\) of \((G, S)\)

- A Salem number is a real algebraic integer \(\alpha\) greater than 1 such that all its conjugates other than \(\alpha\) and \(\alpha^{-1}\) lie on the unit circle.
- A Pisot-Vijayaraghavan number (P.V. number) is a real algebraic integer greater than 1 such that the absolute value of its conjugates other than itself are all less than 1.
- A Perron number is a real algebraic integer \(\alpha\) greater than 1 such that the absolute value of its conjugates other than itself are all less than \(\alpha\).

Theorem 3. (Main theorem (Komori-U.))

\(P \subset \mathbb{H}^n\): a noncompact Coxeter simplex
\((G, S)\): a Coxeter group generated by \(P\)
\(\tau\) is a Perron number,

\(\tau\) is a unique root of degree \(n\) of \(g(t) = \sum_{i=0}^{n} a_i t^i - 1\),
where \(a_n\) is a non-negative integer.

We also assume that the greatest common divisor of \(\{k \in \mathbb{N} | a_k \neq 0\}\) is 1. Then there is a real number \(r_0, 0 < r_0 < 1\) which is a unique zero of \(g(t)\) having the smallest absolute value of all zeros of \(g(t)\).

Conjecture. (Kellerhals-Perren)

\(P \subset \mathbb{H}^n\): a Coxeter polyhedron
\((G, S)\): a Coxeter group generated by \(P\)

\(\tau\) is a Perron number.

References